Invariant Kreĭn subspaces, regular irreducibility and integral representations

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Abstract

We study unitary representations of groups in Kreĭn spaces, irreducibility criteria and integral decompositions. Our main tool is the theory of Kreĭn subspaces and their (reproducing) kernels and a variant of Choquet's theorem.

Keywords Kreĭn space, reproducing kernel, invariant subspace, integral representation

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Introduction

The use of reproducing kernels methods in harmonic analysis and representation theory is now classical [37], [13], [34], [42], [44], and is one of the main applications of the theory of reproducing kernel Hilbert spaces or Hilbert subspaces [5], [36].

The present paper was motivated by the intuition that such methods could also apply to representations in indefinite inner product spaces. Hence we consider together two less known extensions of the previous theories: on one hand, the theory of Hermitian subspaces and Kreĭn subspaces and their kernels [36], [1], [39], and on the other hand operator algebras and group representations in indefinite inner product spaces [31], [24], [21], [32], [3]. Note that these representations appear mainly in mathematical physics, following the use of indefinite

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metric spaces in the works of Dirac [10], Pauli [33] and Heisenberg [17]. They are now critical issues in quantum electromagnetism and the Gupta-Bleuler triplet [4], representations of CCR algebras [29], [30] or the QFT formalism [40] and the study of De Sitter spaces, [14]. Note also that since the pioneering work of Pontryagin [35], the theory of linear operators in Kreĭn spaces has been developed into a major branch of modern operator theory [19], [23], [2].

The paper is organized as follows: in section 1 we review some general facts about Kreĭn spaces, and define Kreĭn subspaces and their kernels. Section 2 is devoted to indefinite representations, and discuss the links between invariant Kreĭn subspaces and their kernels. It gives also criteria for irreducibility and a variant of Schur's lemma. Sections 3 and 4 then discuss the existence of a direct integral decomposition into irreducibles subspaces.

1 Kreĭn spaces, Kreĭn subspaces and kernels

1.1 Kreĭn spaces

A Kreĭn space is an indefinite inner product space $(\mathcal{K}, [\cdot,\, \cdot])$ (*i.e.* the form $[\cdot,\, \cdot]$ is sesquilinear and Hermitian) such that there there exists an automorphism J of $\mathcal{K}$ which squares to the identity (called fundamental symmetry or signature operator), $\langle x,\, y\rangle \equiv [Jx,\, y]$ defines a positive definite inner product and $(\mathcal{K}, \langle\cdot,\, \cdot\rangle)$ is a Hilbert space. Equivalently, the indefinite inner product space $(\mathcal{K}, [\cdot,\, \cdot])$ is a Kreĭn space if there exist an admissible (with respect to the inner product) hilbertian topology on $\mathcal{K}$ that makes it an Hilbert space.

The following subsets are defined in terms of the "square norm" induced by the indefinite inner product:

- $\mathbf{K}_+ \equiv \{x \in \mathcal{K} : [x,\, x] > 0\}$ is called the "positive cone";
- $\mathbf{K}_- \equiv \{x \in \mathcal{K} : [x,\, x] < 0\}$ is called the "negative cone";
- $\mathbf{K}_0 \equiv \{x \in \mathcal{K} : [x,\, x] = 0\}$ is called the "neutral cone".

A subspace $\mathcal{L} \subset \mathcal{K}$ lying within $\mathbf{K}_0$ is called a "neutral subspace". Similarly, a subspace lying within $\mathbf{K}_+$ ($\mathbf{K}_-$) is called "positive" ("negative"). A subspace in any of the above categories may be called "semi-definite", and any subspace that is not semi-definite is called "indefinite".

Any decomposition of the indefinite inner product space $\mathcal{K}$ into a pair of subspaces $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$ such that $\mathcal{K}_+ \subset \mathbf{K}_+ \cup \{0\}$ and $\mathcal{K}_- \subset \mathbf{K}_- \cup \{0\}$ is

called a "fundamental decomposition" of $\mathcal{K}$. $\mathcal{K}_+$ equipped with the restriction of the bilinear form $[\cdot,\cdot]$ is then a Hilbert space, and $\mathcal{K}_-$ the antispace of a Hilbert space $|\mathcal{K}_-|$. To this fundamental decomposition is associated a fundamental symmetry J such that the scalar product $\langle x,\, y\rangle \equiv [Jx,\, y]$ coincide with the scalar product of $\mathcal{H} = \mathcal{K}_+ \oplus |\mathcal{K}_-|$.

The positive definite inner product $\langle\cdot,\,\cdot\rangle$ depends on the chosen fundamental decomposition, which is, in general, not unique. But (see [11]) any two fundamental symmetries J and J' compatible with the same indefinite inner product on $\mathcal{K}$ result in Hilbert spaces $|\mathrm{K}|$ and $|\mathrm{K}'|$ whose decompositions $|\mathrm{K}|_\pm$ and $|\mathrm{K}'|_\pm$ have equal dimensions. Moreover they induce equivalent square norms hence a unique topology. This topology is admissible, and it is actually the Mackey topology defined by the bilinear pairing. All topological notions in a Kreĭn space, like continuity or closedness of sets are understood with respect to this Hilbert space topology.

Orthogonality is a key issue in indefinite inner product spaces. Let $\mathcal{L}$ be a subspace of $\mathcal{K}$. The subspace $\mathcal{L}^{[\perp]} \equiv \{x \in \mathcal{K} : [x,\, y] = 0 \text{ for all } y \in \mathcal{L}\}$ is called the orthogonal companion of $\mathcal{L}$. If J is a fundamental symmetry it is related to the (Hilbert) orthogonal by $\mathcal{L}^{[\perp]} = (J\mathcal{L})^\perp$. $\mathcal{L}^0 \equiv \mathcal{L} \cap \mathcal{L}^{[\perp]}$ is called the isotropic part of $\mathcal{L}$. If $\mathcal{L}^0 = \{0\}$, $\mathcal{L}$ is called non-degenerate. It is called regular (or a Kreĭn subspace) if it is closed and a Kreĭn space with respect to the restriction of the indefinite inner product. This is equivalent to $\mathcal{L} \oplus \mathcal{L}^{[\perp]} = \mathcal{K}$ ([11]) and this relation is sometimes taken as a definition of regular subspaces.

If $\mathcal{K}$ and $\mathcal{H}$ are Kreĭn spaces, then continuity of operators is defined with respect to the Hilbert norm induced by any fundamental decomposition.

Any continuous (weakly continuous) operator A has an adjoint $A^{[*]}$ (with respect to the indefinite inner product) verifying

$$\forall k \in \mathcal{K},\, \forall h \in \mathcal{H}, \qquad [Ak, h]_{\mathcal{H}} = [k, A^{[*]}h]_{\mathcal{K}} \tag{1.1}$$

This adjoint is sometimes called J-adjoint (to emphasize the role of the fundamental symmetry J) and it is related to the classical (Hilbert) adjoint by $A^{[*]} = JA^*J$.

1.2 Kreĭn subspace and kernels

We first make some definitions on remarks on kernels.

Definition 1.1

1. *Let $\mathcal{E}$ be a l.c.s., $\mathcal{E}'$ its topological dual and $\overline{\mathcal{E}'}$ the conjugate space of its topological dual. Then according to L. Schwartz [36], we call kernel any weakly continuous linear application $\varkappa : \overline{\mathcal{E}'} \longrightarrow \mathcal{E}$.*

2. *More generally, if $(\mathcal{E}, \mathcal{F})$ is a semi-duality (a pair $(\mathcal{E}, \mathcal{F})$ of spaces together with a non-degenerate sesquilinear form $(.,.)_{(\mathcal{F},\mathcal{E})}$), we call kernel any weakly continuous linear application $\varkappa : \mathcal{F} \longrightarrow \mathcal{E}$.*

Since $\mathcal{F}$ is identified with the space of continuous semilinear forms on $\mathcal{E}$, the adjoint of $\varkappa$ ($\varkappa^*$) is also a kernel of $(\mathcal{E}, \mathcal{F})$. The kernel $\varkappa$ is Hermitian if $\varkappa^* = \varkappa$. It is positive if

$$\forall \varphi \in \mathcal{F}, (\varphi, \varkappa(\varphi))_{(\mathcal{F},\mathcal{E})} \geq 0 \tag{1.2}$$

A positive kernel is Hermitian. We note the set of positive kernels $\mathbf{L}^+(\mathcal{F}, \mathcal{E})$, the set of Hermitian kernels $\mathbf{L}^h(\mathcal{F}, \mathcal{E})$.

It is crucial to note that the set of positive kernels $\mathbf{L}^+(\mathcal{F}, \mathcal{E})$ is a salient convex cone (the positivity conditions then defines a partial order on the set of positive kernels: $H \leq K \iff K - H \geq 0$.)

The set of Hermitian kernels will usually be to large for our applications and we will mainly study bounded kernels. A Hermitian kernel K is bounded by a positive kernel H if $H - K \geq 0$ and $H + K \geq 0$. In this case H is called a majorant of K and we say that (K, H) is a bounded pair (of kernels). We note the set of bounded hermitian kernels $\mathbf{L}^b(\mathcal{F}, \mathcal{E})$.

We say that positive kernels K and L are independent if the following statement holds:
If $0 \leq H \leq K$ and $0 \leq H \leq L$ then $H = 0$.
We will sometimes use the following notation: if K and L are positive (kernels $K + L = K \oplus L$ means that the kernels are independent.

Definition 1.2 *(K, H) is a Kolmogorov Hermitian pair (or minimal pair) of kernels if K is a Hermitian kernel, H a positive kernel that bounds K and $H - K, H + K$ are independent.*

Second, we define Hilbert and Kreĭn subspaces of a semi-duality $(\mathcal{E}, \mathcal{F})$ (equivalently of a l.c.s. $\mathcal{E}$).

Definition 1.3

1. *A Hilbert subspace $\mathcal{H}$ of $(\mathcal{E}, \mathcal{F})$ is a Hilbert space continuously (for the Mackey topologies) included in $\mathcal{E}$.*

2. *A Hermitian subspace $\mathcal{K}$ of $(\mathcal{E}, \mathcal{F})$ is the difference of two (disjoint) Hilbert subspaces of $(\mathcal{E}, \mathcal{F})$.*
3. *A Kreĭn subspace $\mathcal{K}$ of $(\mathcal{E}, \mathcal{F})$ is a Kreĭn space continuously (for the Mackey topologies) included in $\mathcal{E}$.*

It si straightforward to see that the last two notions coincide (for details on Hermitian subspaces an Kreĭn subspaces, we refer to [25]). We note $Hilb((\mathcal{E}, \mathcal{F}))$ the set of Hilbert subspaces of $(\mathcal{E}, \mathcal{F})$ and $Krein((\mathcal{E}, \mathcal{F}))$ the set of Kreĭn subspaces of $(\mathcal{E}, \mathcal{F})$.

We can now state the main results of the theory of Hilbert subspaces and Kreĭn subspaces:

Theorem 1.4 *Suppose $\mathcal{E}$ is quasi-complete (for its Mackey topology). Then there is a bijection between $Hilb((\mathcal{E}, \mathcal{F}))$ and $\mathbf{L}^+(\mathcal{F}, \mathcal{E})$. Moreover, this bijection is an isomorphism of convex cones.*

Any reader particularly interested by the isomorphism of convex cone structure can read [36] p 159-161, where the proof is detailed. The kernel H of a Hilbert subspace $\mathcal{H}$ is characterized by the following equality:

$$\forall \varphi \in \mathcal{F},\ \forall h \in \mathcal{H}, \quad \langle H(\varphi), h\rangle_{\mathcal{H}} = (\varphi, h))_{\mathcal{F},\mathcal{E}} \tag{1.3}$$

Theorem 1.5 *There is a surjective morphism of convex cones between $Krein((\mathcal{E}, \mathcal{F}))$ and $\mathbf{L}^b(\mathcal{F}, \mathcal{E})$. This is* ***not*** *an isomorphism in general.*

The kernel K of a Kreĭn subspace $\mathcal{K}$ is characterized by an equation similar to 1.3:

$$\forall \varphi \in \mathcal{F},\ \forall k \in \mathcal{K}, \quad [K(\varphi), k]_{\mathcal{K}} = (\varphi, k))_{\mathcal{F},\mathcal{E}} \tag{1.4}$$

The existence of so-called kernels of multiplicity is the major difference between the two theories. It follows that all the constructions that rely uniquely on the kernel may fail, as we will see studying integral of Kreĭn subspaces. To circumvent this flaw we introduce the notion of Kreĭn-Hilbert pairs.

1.3 Kreĭn-Hilbert pairs

Definition 1.6 *A pair $(\mathcal{K}, \mathcal{H})$ is called a Kreĭn-Hilbert pair $\mathcal{K}$ is a Kreĭn space continuously included in the Hilbert space $\mathcal{H}$.*
It is called a closed Kreĭn-Hilbert pair if $\mathcal{K}$ is a Kreĭn space, closed subspace of

the Hilbert space $\mathcal{H}$.
It is a fundamental (or minimal) pair if there exists two Hilbert spaces $\mathcal{H}_+$ and $\mathcal{H}_-$,

$$\mathcal{K} = \mathcal{H}_+ \ominus \mathcal{H}_- \text{ and } \mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$$

Of course any fundamental pair is closed. Any pair $(\mathcal{K}, \mathcal{H})$ define a (bounded) hermitian kernel $\chi : \mathcal{H} \to \mathcal{H}$. It is interesting at this point to note that even in this case the kernel χ may be of multiplicity (see [36], [16], [9]).

Example 1.7 *Let $(\mathcal{K}, J)$ be a Kreĭn space with fundamental symmetry J. Then $\mathcal{H} = \mathcal{K}$ endowed with the scalar product $\langle k, h\rangle_{\mathcal{H}} = [k, Jh]_{\mathcal{K}}$ is a a Hilbert space and $(\mathcal{K}, \mathcal{H})$ is a fundamental Kreĭn-Hilbert pair.*

We note $KH((\mathcal{F}, \mathcal{E}))$ the set of minimal Kreĭn-Hilbert pairs of subspaces $(\mathcal{K}, \mathcal{H})$, equivalently of pairs $(\mathcal{K}, \mathcal{H})$ of a Kreĭn subspace and a Hilbert subspace of $\mathcal{E}$, such that their kernels (K, H) form a minimal pair.

1.4 Image by a weakly continuous application

We suppose now we are given a second pair of spaces in duality $(\mathfrak{E}, \mathfrak{F})$. It is actually possible ([36], [26] and proofs therein) to define the image of a Kreĭn space $\mathcal{K}$ by a weakly continuous linear application $u : \mathcal{E} \to \mathfrak{E}$ by using orthogonal relations in the duality $\mathcal{K}$, but this image is not in general a Kreĭn space.
We recall this construction hereafter. $\forall A \subset \mathcal{E},\ u_{|A}$ denotes the restriction of u to the set A. We then define the following quotient space:

$$\mathcal{M} = \left(\ker(u_{|\mathcal{K}})^{[\perp]} / \ker(u_{|\mathcal{K}})\right)$$

Lemma 1.8 *The linear application $u_{|\mathcal{M}}$ is well defined and injective, and $\forall\,(\dot{m}, \dot{n}) \in \mathcal{M}^2$, the bilinear form $B(u_{|\mathcal{M}}(\dot{n}), u_{|\mathcal{M}}(\dot{m})) = [n, m]_{(\mathcal{K})}$ defines a indefinite inner product on the space $u_{|\mathcal{M}}(\mathcal{M})$.*

Proof We have the following factorisation

$$u : \ker(u_{|\mathcal{K}})^{[\perp]} \longrightarrow (\ker(u_{|\mathcal{K}})^{[\perp]} / \ker(u_{|\mathcal{K}}) \xrightarrow{u_{|\mathcal{M}}} \mathfrak{E}$$

and $u_{|\mathcal{M}}$ is one-to-one. Moreover the bilinear form $B : u_{|\mathcal{M}}(\mathcal{M}) \times u_{|\mathcal{M}}(\mathcal{M}) \longrightarrow \mathbb{K}$ is well defined since:
$\forall (m_1, m_2) \in \dot{m},\ \forall (n_1, n_2) \in \dot{n},\ [m_1 - m_2, n_1 - n_2]_{\mathcal{K}} = 0.$

However, it may happen that this image is Mackey-complete (for instance if u is one-to-one, or more generally if $\ker(u)$ is regular). In this case the space $u_{|\mathcal{M}}(\mathcal{M})$ is actually a Kreĭn space continuously embedded in $\mathfrak{E}$, and we can compute its kernel:

Theorem 1.9
If $u_{|\mathcal{M}}(\mathcal{M})$ is a Kreĭn space, then it is a Kreĭn subspace of $\mathfrak{E}$. Its kernel is $u \circ \varkappa \circ^t u$.

1.5 Integral of Kreĭn subspaces: the neccessity of Kreĭn-Hilbert pairs

The theory of direct integral of Hilbert spaces is well known, as is the theory of integral of Hilbert subspaces ([36], [42]), and poses no difficulties. This is not the case for Kreĭn spaces, where it is actually not possible to define directly the direct integral of Kreĭn spaces as the following example shows:

Example 1.10 *Define on $\mathbb{R}^2$ the following inner products :*

$$\begin{aligned} \left[\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} | \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right] &= x_1 y_2 + x_2 y_1 \\ \langle \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} | \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \rangle_n &= \frac{1}{n^2} x_1 x_2 + n^2 y_1 y_2 \end{aligned}$$

Then $\left(\mathcal{K} = \mathbb{R}^2, [.|.]\right)$ is a Kreĭn space, and

$$J_n = \begin{pmatrix} 0 & n^2 \\ \frac{1}{n^2} & 0 \end{pmatrix}, \ n \in \mathbb{N}^*$$

are fundamental symmetries associated with the spaces $\left(\mathcal{H}_n = \mathbb{R}^2, \langle .|. \rangle_n\right)$.

We want to give a sense to $\mathfrak{K} = \bigoplus_{n=1}^{+\infty} \mathcal{K}$. Fact is that there are many possible interpretations of this space in terms of Kreĭn space. For instance

$$\mathfrak{K}_1 = \mathfrak{H}_1 = \bigoplus_{n=1}^{+\infty} \mathcal{H}_1 = \{h = k_1 \oplus k_2 \oplus ..., \ \sum_{n=1}^{\infty} ||k_n||_1^2 < +\infty\}$$

as a vector space, with inner product

$$\left[k = \sum_{n=1}^{+\infty} k_n, h = \sum_{n=1}^{+\infty} h_n\right] = \sum_{n=1}^{+\infty} [k_n, h_n]_{\mathcal{K}}$$

(one checks easily that this inner product is well defined for elements of $\mathfrak{H}_1$.)
But

$$\mathfrak{K}_{\mathbb{N}} = \mathfrak{H}_{\mathbb{N}} = \bigoplus_{n=1}^{+\infty} \mathcal{H}_n = \{h = k_1 \oplus k_2 \oplus ..., \sum_{n=1}^{\infty} ||k_n||_n^2 < +\infty\}$$

as a vector space, with inner product

$$\left[k = \sum_{n=1}^{+\infty} k_n, h = \sum_{n=1}^{+\infty} h_n\right] = \sum_{n=1}^{+\infty} [k_n, h_n]_{\mathcal{K}}$$

is a second valuable choice, distinct from the first since $e = e_1 \oplus e_1 \oplus e_1...$ is in $\mathfrak{H}_{\mathbb{N}}$ but not in $\mathfrak{H}_1$.
In terms of kernels, this has the following interpretation: Let $\mathcal{E} = \mathbb{R}^{\mathbb{N}\times\{1,2\}}$ endowed with the topology of pointwise convergence. $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are Hilbert subspaces of $\mathcal{E}$, and $\mathfrak{K}_1$ and $\mathfrak{K}_2$ are distinct Kreĭn subspaces of $\mathcal{E}$ but with the same kernel (that is a kernel of multiplicity)

$$K((n,.),(m,.)) = \delta_{n,m} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Note that by corollary 2 p.253 in [36], $\mathcal{E}$ being the dual of a barreled nuclear space, this kernel is automatically of multiplicity.

The idea is then to work on fundamental Kreĭn-Hilbert pairs.

Let T be a locally compact space endowed with a measure m. Let $t \to (\mathcal{K}_t, \mathcal{H}_t)$ an application from T to $KH((\mathcal{F}, \mathcal{E}))$, (K_t, H_t) be the kernels of $(\mathcal{K}_t, \mathcal{H}_t)$. We say that the family $\{(\mathcal{K}_t, \mathcal{H}_t),\ t \in T\}$ is pseudo m-integrable if for all $\phi \in \mathcal{F}$, the function $t \to (\phi, H_t(\phi))_{(\mathcal{F},\mathcal{E})}$ is integrable with respect to m. The integral will be constructed as follows:

Let

$$\Pi\mathcal{H} = \left\{\{h_t \in \mathcal{H}_t,\ t \in T\},\ \int_T ||h_t||^2_{\mathcal{H}_t} < +\infty\right\}/\mathcal{R} \tag{1.5}$$

where $\mathcal{R}$ is the equivalence relation of m almost sure equality, with norm

$$||\{h_t \in \mathcal{H}_t,\ t \in T\}||^2_{\Pi\mathcal{H}} = \int_T ||h_t||^2_{\mathcal{H}_t} \tag{1.6}$$

This norm makes $\Pi\mathcal{H}$ a Hilbert space. Let $\Pi\mathcal{K} = \Pi\mathcal{H}$ as a vector space and endow it with the indefinite form

$$\left[\{k_t \in \mathcal{K}_t,\ t \in T\}, \left\{k_t^{'} \in \mathcal{K}_t,\ t \in T\right\}\right]_{\Pi\mathcal{K}} = \int_T \left[k_t, k_t^{'}\right]_{\mathcal{K}_t} dm(t) \tag{1.7}$$

Then $\Pi\mathcal{K}, [.,.]_{\Pi\mathcal{K}}$ is a Kreĭn space.

From [36] there exists a continuous linear application Φ from $\Pi\mathcal{H}$ to $\mathcal{F}^*$ (algebraic dual of $\mathcal{F}$ endowed with the topology $\sigma(\mathcal{F}^*, \mathcal{F})$) defined by:

$$\Phi(\{h_t \in \mathcal{H}_t,\, t \in T\}) = \int_T h_t dm(t) \tag{1.8}$$

where the second member is understood as the weak integral of a scalarly integrable function. From the general theory of Hilbert subspaces [36] and subdualities [26] (and previous section) the image of the Hilbert space $\Pi\mathcal{H}$ is the Hilbert subspace $\int_T \mathcal{H}_t dm(t)$ of $\mathcal{F}^*$ with kernel $H = \int_T H_t dm(t) \in \mathcal{L}^+(\overline{\mathcal{F}}, \mathcal{F}^*)$, and the image of $\Pi\mathcal{K}$ is a self-subduality (pseudo-Kreĭn subspace in the terminology of [18]) $\int_T \mathcal{K}_t dm(t)$ of $\mathcal{F}^*$ with kernel $H = \int_T K_t dm(t)$. Remark that $\int_T \mathcal{K}_t dm(t) \subset \int_T \mathcal{H}_t dm(t)$ but is not equal in general.

If the space $\int_T \mathcal{K}_t dm(t)$ is a Kreĭn space we say that the family is m-integrable. Remark that this is equivalent with saying that $\left(\int_T \mathcal{K}_t dm(t), \int_T \mathcal{H}_t dm(t)\right)$ is a Kreĭn-Hilbert pair. If Φ is one-to-one, then the family is actually m-integrable ($\left(\int_T \mathcal{K}_t dm(t), \int_T \mathcal{H}_t dm(t)\right)$ is a fundamental Kreĭn-Hilbert pair) and in this case we say that the integral is direct.

Under these assumption, the norm in $\int_T^{\oplus} \mathcal{H}_t dm(t)$ is just

$$|| \int_T^{\oplus} h_t dm(t) ||^2_{\int_T^{\oplus} \mathcal{H}_t dm(t)} = \int_T ||h_t||^2_{\mathcal{H}_t} dm(t) \tag{1.9}$$

and the indefinite inner product in $\int_T \mathcal{K}_t dm(t)$ is

$$\left[\int_T^{\oplus} k_t dm(t), \int_T^{\oplus} k_t^{'} dm(t)\right]_{\int_T^{\oplus} \mathcal{K}_t dm(t)} = \int_T \left[k_t, k_t^{'}\right]_{\mathcal{K}_t} dm(t) \tag{1.10}$$

2 Group representations in Kreĭn subspaces

2.1 Unitary representations and Hilbert subspaces

The study of group representation in Hilbert subspaces is one of the many approaches to harmonic analysis. The usual setting is the study representations of a Lie group G on Hilbert subspaces (for instance $L^2(\mu)$ for an invariant measure on G) of the space of distributions on $X = G$ (or X an homogeneous space [7], [42]). One can also study Hilbert subspaces of other locally convex

spaces, such as spaces of holomorphic functions [13], [12].
We refer to the works [8], [22], [6] for fundamentals theorems on irreducibility in reproducing kernel Hilbert spaces or Hilbert subspaces.

2.2 Extension to Kreĭn subspaces

In this section $T : \mathcal{E} \to \mathcal{E}$ is a weakly continuous operator, and G is a group of weakly continuous endomorphisms of $\mathcal{E}$ (in this section we identify a group G with is image under a given representation τ in $\mathcal{L}(\mathcal{E})$).

Definition 2.1 *A Kreĭn subspace $\mathcal{K}$ (with kernel K) is invariant under T if $T(\mathcal{K}) \subset \mathcal{K}$, and T is a unitary operator ($T^{[*]}T = TT^{[*]} = I_{\mathcal{K}}$).*

Example 2.2 *Let $\theta \in \mathbb{R}$, and define $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ by*

$$T_\theta = \begin{pmatrix} i\sinh(\theta) & \cosh(\theta) \\ \cosh(\theta) & i\sinh(\theta) \end{pmatrix}$$

Then the Kreĭn space $\mathcal{K}$ of example 1.10 is invariant with respect to T_θ.

Proposition 2.3 *A Kreĭn subspace $\mathcal{K}$ (with kernel K) is invariant under T if $T(\mathcal{K}) \subset \mathcal{K}$, $TKT^* = K$ and $T(\mathcal{K})$ is a Kreĭn space.*

Proof Let $T(\mathcal{K})$ be the inner product space, image of the Kreĭn space $\mathcal{K}$. Since it is Kreĭn space, by theorem 1.9 it is a Kreĭn subspace of $\mathcal{E}$ with kernel is $TKT* = K$. But $T(\mathcal{K}) \subset \mathcal{K}$, and by proposition 39 in [36] the two Kreĭn spaces coincide, and T acts as a unitary operator on $\mathcal{K}$.

Of course, if $\mathcal{K}$ is finite-dimensional then the equality $TKT^* = K$ is sufficient.

Definition 2.4 *A Kreĭn subspace $\mathcal{K}$ (with kernel K) is invariant under G if $\forall g \in G,\ g(\mathcal{K}) \subset \mathcal{K}$, $gKg^* = K$.*

Remark 2.5 *Since G is a group, then $g(\mathcal{K})$ is always a Kreĭn space.*

Theorem 2.6 *A Kreĭn subspace $\mathcal{K}$ (with kernel K and minimal bound H) is invariant under a group G if and only if*

1. $\forall g \in G,\ gKg* = K$
2. $\forall g \in G,\ \exists \alpha_g,\ gHg* \leq \alpha_g H$

Proof Let $g \in G$. By proposition 2.3 and remark 2.5, $\mathcal{K}$ is invariant under g if $g(\mathcal{K}) \subset \mathcal{K}$ or equivalently, if $g(\mathcal{H}) \subset \mathcal{H}$. But $g(\mathcal{H})$ is a Hilbert subspace of $\mathcal{E}$ with kernel gHg^*, and by proposition 15 in [36], the inclusion holds if and only if exits $c > 0,\ gHg^* \leq cH$.

Definition 2.7 *A invariant Kreĭn subspace $\mathcal{K}$ is called (topologically) irreducible if it admits no invariant closed subspace apart from 0 and $\mathcal{K}$.*
It is called indecomposable if any decomposition as a direct sum of two invariant closed subspaces involves the trivial subspaces.

Definition 2.8 *A invariant Kreĭn subspace $\mathcal{K}$ is called regularly irreducible if it admits no invariant regular subspace (closed and a Kreĭn space with the induced sesquilinear form) apart from 0 and $\mathcal{K}$.*
It is called regularly indecomposable if any decomposition as a direct sum of two invariant regular subspaces involves the trivial subspaces.

Finally, following Kissin[21], we say that a representation on $\mathcal{K}$ is non-degenerate if $\mathcal{K}$ has no neutral invariant subspace.

From [37] we have:

Lemma 2.9 *For a Hilbert space, the two notions of irreducibility coincide.*

Theorem 2.10 *For Krein spaces:*

- *Irreducibility $\Rightarrow$ indecomposability, but the converse is not true.*
- *Regular irreducibility $\iff$ regular indecomposability.*

Proof The implication is straightforward.
For the converse, suppose that $\mathcal{K}$ is indecomposable, and let $\mathcal{H}$ be a regular subspace of $\mathcal{K}$. Then $\mathcal{H}^\perp$ is a regular subspace and $\mathcal{H} \oplus \mathcal{H}^\perp = \mathcal{K}$. By indecomposability, $\mathcal{H} = 0$ or $\mathcal{H} = \mathcal{K}$.

2.3 Schur's lemma in Kreĭn spaces

A main tool in harmonic analysis is Schur's lemma, that asserts that for unitary representations in Hilbert spaces irreducibility is equivalent with operator irreducibility. The aim of this section is to study the link between the different notions in the Kreĭn space setting. Note that a strange phenomenon will occur, for in the algebra of bounded operators on a Kreĭn space the Gelfand-Naimark property is not valid [27], and there exists self-adjoint nilpotent operators.

Also, the set of self-adjoint operators in a Kreĭn space is too large for a good spectral theory. Hence it is classical to study definitizable operators [20], [23]. An operator A is definitizable if there exists a polynomial p such that $p(A)$ is a positive operator. Hence we start with the study of positive operators in the commutant of G.

Lemma 2.11 *Let $\mathcal{K}$ be a Kreĭn space, regularly irreducible under a the action of a group of unitary operators G.*
Let A be a bounded positive operator and suppose that

$$\forall T \in G, ,\ TA = AT$$

Then either $\mathcal{K}$ is a Hilbert space and $A = \lambda I,\ \lambda \geq 0$ or $\mathcal{K}$ is not definite and $A = N;\ N nilpotent\ and\ positive, N^2 = 0$.

If moreover the representation is non-degenerate then $A = 0$.

Proof If $\mathcal{K}$ is a Hilbert space then it is Schur's lemma.
Suppose now that $\mathcal{K}$ is not definite. Since A is positive we have a spectral decomposition [23]. We note $\mathcal{K}_\rho$ the corresponding spectral subspaces. These are Kreĭn spaces and there orthogonal sum is $\mathcal{K}$. Moreover, these spaces are invariant since spectral projections are in the double commutant of A.
It follows that all of the spectral subspaces are $\{0\}$ except one that is $\mathcal{K}$. By spectral decomposition, $A - \rho I$ is nilpotent, $A = \rho I + N$. Suppose $\rho \neq 0$. Then $B = I + \rho^{-1}N = I + M$ is positive and invertible, $B^{-1} = I - M$ is also positive and $B + B^{-1} = 2I$ is positive, which is excluded by hypothesis. It follows that $\rho = 0$, and N is positive, $N^2 = 0$.

We can now state a general result for definitizable operators in the commutant of G:

Lemma 2.12 (Regular Schur's lemma for Kreĭn spaces) *Let $\mathcal{K}$ be a Kreĭn space, regularly irreducible under a the action of a group of unitary operators*

G.
Let A be a bounded definitizable (self-adjoint) operator and suppose that

$$\forall T \in G,\ TA = AT$$

Then

$$A = N + \lambda I;\ N \textit{nilpotent}.$$

If moreover the representation is non-degenerate then $A = \lambda I$.

Proof Since A is definitizable we have a spectral decomposition. As before, by irreducibility all of the spectral subspaces are $\{0\}$ except one that is $\mathcal{K}$, and $A - \rho I$ is nilpotent.

Suppose now that the representation is non-degenerate. Let N be nilpotent of order $k > 1$ and pose $M = N^{k-1}$. Then $M^2 = 0$, $\overline{ImM}$ is neutral and invariant hence 0 or $\mathcal{K}$, absurd. Then $k = 1$ and $N = 0$.

Example 2.13 *Let $\mathcal{E} = \mathcal{H} \oplus \mathcal{H}$ where $\mathcal{H}$ is a Hilbert space and*

$$G = \{\begin{pmatrix} I & B \\ 0 & I \end{pmatrix}\}$$

with B anti-symmetric. Remark that G is a (multiplicative) group. Suppose also that an invertible antisymmetric operator exists.

Pose $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then G defines J-unitary operators, $Q = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is nilpotent and commutes with G. Remark that $\mathcal{M} = (\mathcal{H}, 0)$ is invariant under G, neutral in $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}$ and $\mathcal{K}$ is regularly irreducible, since only trivial (J-self-adjoint) projection commute with G by the existence of the invertible and antisymmetric operator.

2.4 "Fundamental" representations

It may happen that the group G carries a fundamental symmetry of $\mathcal{K}$:

$$\exists g \in G,\ \mathrm{g} = g^{[*]} = g^{-1} \text{ and } g \text{ positive.} \tag{2.1}$$

In this case, we say that the representation is fundamental. The interest of fundamental representations lies in the following lemma and theorem:

Lemma 2.14 (Regular Schur's lemma for fundamental representation) *Let $\mathcal{K}$ be a Kreĭn space, regularly irreducible under the "fundamental" action of a group of unitary operators G.*
Let A be a bounded operator and suppose that

$$\forall T \in G,\ TA = AT$$

Then

$$A = \lambda I$$

Proof Let J be a fundamental symmetry associated to G. Pose $B = \frac{A+A^{[*]}}{2}$, $C = i\frac{A-A^{[*]}}{2}$. Since G acts by unitary operators, $A^{[*]}$ also commutes with G and B, C are self-adjoint and commute with G. But they also commute with a fundamental symmetry, hence they are self-adjoint in the Hilbert space sense, admit a spectral function and by irreducibility their spectrum reduces to a single number. Finally $A = \lambda I$.

Theorem 2.15 *For fundamental representations, regular irreducibility implies (topological) irreducibility.*

Proof Let $\mathcal{L}$ be a closed subspace invariant with respect to the fundamental symmetry $J \in G$. Then $J\mathcal{L} = \mathcal{L}$, and it follows that $\mathcal{L}^{[\perp]} = (J\mathcal{L})^{\perp} = \mathcal{L}^{\perp}$. But $\mathcal{L} \oplus \mathcal{L}^{\perp} = \mathcal{K}$ and $\mathcal{L}$ is regular, hence trivial.

As a consequence:

Corollary 2.16 *For fundamental representations, the three following notions coincide:*

1. *Regular irreducibility;*
2. *Operator irreducibility;*
3. *Topological irreducibility.*

Note that a fundamental representation is then obviously non-degenerate.

2.5 Reproducing kernel Kreĭn space and irreducible representations

In this section we suppose that $\mathcal{E} = \mathbb{C}^X$, where X is a set, and G is a group acting transitively on X (for instance X is an homogeneous space). There is a

canonical action of G on $\mathcal{E}$ defined by:

$$\forall g \in G,\ \tau_g(f)(x) = f(g^{-1}(x)) \tag{2.2}$$

Let $\mathcal{K}$ be a Kreĭn subspace of $\mathcal{E}$ (we call such a subspace a reproducing kernel Kreĭn space) invariant with respect to τ.

Note that for such a subspace [36], [28], [26] we can identify its (unique) kernel with a reproducing function $K(.,.)$ on X^2 that verifies:

$$\forall (x,y) \in X^2,\ [K(x,.), K(y,.)]_{\mathcal{K}} = K(x,y) \tag{2.3}$$

or equivalently (this is equation 1.4)

$$\forall x \in X,\ \forall k \in \mathcal{K},\ [K(x,.), k]_{\mathcal{K}} = k(x) \tag{2.4}$$

For any $\omega \in X$, we define its isotropy group $\varpi = \{g \in G,\ g\omega = \omega\}$. For such a subgroup we define the subspace $\mathcal{K}^{\varpi} = \{k \in \mathcal{K},\ \forall \varrho \in \varpi,\ \tau_{\varrho}(k) = k\}$ of ϖ-invariant functions.

Theorem 2.17

1. *if K is the reproducing kernel function of $\mathcal{K}$,*

$$\forall g \in G,\ \forall (x,y) \in X^2,\ K(gx, gy) = K(x,y) \tag{2.5}$$

2. *if $\mathcal{K} \neq \{0\}$ then $\mathcal{K}^{\varpi} \neq \{0\}$.*

3. *if $\dim(\mathcal{K}^{\varpi}) = 1$ then the representation is regularly irreducible.*

Proof

1. Fix $g \in G$ and define $R(x,y) = K(gx, gy)$. Since the representation is unitary,

$$\begin{aligned} [R(x,.), k]_{\mathcal{K}} &= [K(gx, g.), k]_{\mathcal{K}} = [\tau_g(K(gx, g.)), \tau_g(k)]_{\mathcal{K}} \\ &= \left[K(gx,.), k(g^{-1}.)\right]_{\mathcal{K}} = k(g^{-1}gx) \\ &= k(x) \end{aligned}$$

   and we conclude by unicity of the kernel.

2. Since $\mathcal{K} \neq \{0\}$, the function $K(.,.)$ is not identically zero and exists x, y in X^2, $K(x,y) \neq 0$. Since G acts transitively on X, exists $g \in G,\ gx = \omega$ and by equation 2.5 $K(\omega, gy) \neq 0$. But still by equation 2.5 and the definition of ϖ the function $k(.) = K(\omega,.)$ is in $\mathcal{K}^{\varpi}$, and $\mathcal{K}^{\varpi} \neq \{0\}$.

3. Let $\mathcal{K}_0$ be a regular subspace of $\mathcal{K}$. Then it is a Kreĭn space continuously included in $\mathbb{C}^X$ hence it admits a reproducing kernel function K_0. But K_0 is then G-invariant and $K_0(\omega,.) \in \mathcal{K}^{\varpi}$. Since $\dim(\mathcal{K}^{\varpi}) = 1$ $K_0(\omega,.)$ is proportional to $K(\omega,.)$ and by transitivity of G, K_0 is proportional to K:
$$\exists c \in \mathbb{C},\ \forall x, y \in X^2,\ K_0(x,y) = cK(x,y) \tag{2.6}$$
Now let $k_0 \in \mathcal{K}_0,\ k_0(x) \neq 0$.
$$\begin{aligned} k_0(x) &= [K_0(x,.), k_0(.)]_{\mathcal{K}} \\ &= \overline{c}\,[K(x,.), k_0(.)]_{\mathcal{K}} \\ &= \overline{c}k_0(x) \end{aligned}$$
and $c = 1$. The two Kreĭn subspaces have the same kernel, one is included in tho other hence by proposition 39 in [36] they coincide as Kreĭn spaces.

Example 2.18 (Homogeneous polynomial representations of the Lorentz group) *Let $X = \mathbb{R}^3$ and $G = SO(1,2)$ be the associated Lorentz group. The action of G on X is transitive, and if $J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ is the Minkowski metric operator, then the reproducing kernel $K(x,y) = (\langle Jx, y\rangle_{\mathbb{R}^3})^n$ is invariant under G and it defines a finite-dimensional Kreĭn space of homogeneous polynomials of degree n.*
Let $\omega = e_3$. Then its isotropy subgroup ϖ contains the Lorentz boosts
$$T_\theta = \begin{pmatrix} \cosh(\theta) & \sinh(\theta) & 0 \\ \sinh(\theta) & \cosh(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
It follows the the space of ϖ invariant functions is at most one-dimensional:
Let k be an invariant homogeneous polynomial of degree n. Then it is of the form
$$k(x) = \sum_{i \in I} \alpha_i(\langle, Jy_i, x\rangle)^n \tag{2.7}$$
and its invariance under the Lorentz boosts T_θ implies
$$\forall i \in I,\ T_\theta y_i = y_i$$
Finally $y_i \propto (0,0,1)$ and k is of the form $k(x_1,x_2,x_3) = a(x_3)^n$.
But $\mathcal{K}^{\varpi}$ is also at least of degree one by theorem 2.17 (2), and by (3) $\mathcal{K}$ is regularly irreducible. Note that the representation is actually fundamental ($J \in G$), and $\mathcal{K}$ is topologically irreducible.

3 Integral decomposition in convex cones

In this section, any convex cone Δ will induce its proper order $\leq_\Delta$:

$$(d, d') \in \Delta^2,\ d' \leq_\Delta d \iff d - d' \in \Delta \tag{3.1}$$

Two elements d, d' will be called Δ-independent if $h \leq_\Delta d$, $h \leq_\Delta d'$ implies $h = 0$ and we note $d \amalg_\Delta d'$. This is the relation we definition we used for the cone of positive kernels.

3.1 Integral representation property for closed convex cones

First we recall the definition of the integral representation property (I.R.P.), and the main theorem of integral representation in conuclear cones due to Thomas [41]. In the following $ext(\Gamma)$ denotes the set of extreme rays of any closed convex cone Γ, and $\mathcal{M}^+(X)$ the set of positive radon measures on the topological set X.

Definition 3.1 *A closed convex cone Γ has the I.R.P. if*

1. *for every closed convex subbcone $\Gamma_1 \subset \Gamma$, the map $r : \mathcal{M}^+(ext(\Gamma_1)) \to \Gamma_1$ is onto;*
2. *the map $r : \mathcal{M}^+(ext(\Gamma_1)) \to \Gamma_1$ is bijective if and only if Γ_1 is a lattice (with respect to its proper order).*

Theorem 3.2 *Let F be a weakly complete conuclear space. Then any salient and closed convex cone $\Gamma \subset F$ has the I.R.P.*

For the rest of this section we suppose that $F = \mathfrak{F} \times \mathfrak{F}$ is a weakly complete conuclear space, $\mathfrak{C}$ is a salient closed convex cone of $\mathfrak{F}$ and $\mathfrak{V} = \mathfrak{C} - \mathfrak{C}$ is the vector space generated by $\mathfrak{C}$. It follows that $F = \mathfrak{V} \times \mathfrak{C}$ is a salient closed convex cone of F that has the I.R.P. (for our application, $\mathfrak{F}$ will be the space of kernels and $\mathfrak{C}$ the cone of positive kernels) and every closed convex subcone (for instance the pairs of invariant kernels) will have the I.R.P.

Let $D = \{(v, c) \in \mathfrak{V} \times \mathfrak{C},\ -c \leq_{\mathfrak{C}} v \leq_{\mathfrak{C}} c\}$ be the closed convex cone of dominated pairs. We now define the set of minimal pair in this setting:

Definition 3.3 *The pair $(v, c) \in D$ is $\mathfrak{C}$-minimal (or a minimal pair) if $c - v$ and $c+v$ are independent for the order induced by $\mathfrak{C}$ ($c - v \amalg_{\mathfrak{C}} c + v$), equivalently if any $h \in \mathfrak{C}$ verifying $h \leq_{\mathfrak{C}} c - v,\ h \leq_{\mathfrak{C}} c + v$ is zero.*

Lemma 3.4 *Let $(v, c) \in D$ be a minimal pair. If $(w, d) \leq_D (v, c)$, then (w, d) is $\mathfrak{C}$-minimal.*

Proof Let $(u, p) = (v - w, c - d) \in D$. Let $h \leq_{\mathfrak{C}} d - w,\ h \leq_{\mathfrak{C}} d + w$. Then

$$h - d - p \leq_{\mathfrak{C}} w + u \leq_{\mathfrak{C}} (d - h) + p \tag{3.2}$$

and since $w + u = v,\ d + p = c$, $h - c \leq_{\mathfrak{C}} v \leq_{\mathfrak{C}} c - h$ which gives

$$h \leq_{\mathfrak{C}} v + c,\ h \leq_{\mathfrak{C}} c - v \tag{3.3}$$

and $h = 0$ since (v, c) is minimal.

Lemma 3.5 *Let $(v, c) \in D$ not be minimal. Then there exists $h \in \mathfrak{C} - \{0\}$ and $(v, d) \in D$, $(v, c) = (v, d) + (0, h)$*

Let Γ be a closed convex subcone of $\mathcal{F}$ and $(v, c) \in \Gamma_D = \Gamma \cap D$ be a minimal pair. By the previous theorems if $t \to (e_t, f_t)$, $T \to ext(\Gamma_D)$ is an admissible parametrization of the extreme rays then there exists a Radon measure m on T (unique if the face $\Gamma_D\ ((v, c))$ is a lattice) such that

$$(v, c) = \int_T (e_t, f_t) dm(t) \tag{3.4}$$

Lemma 3.6 *The set of $(t, t') \in T^2$ such that $(e_t + e_{t'}, f_t + f_{t'})$ is not a minimal pair is of m measure 0.*

Proof Suppose m is of mass one. We have

$$(v, c) = \int_T (e_t, f_t) dm(t) = \int_{T^2} (e_t + e_{t'}, f_t + f_{t'}) d(m \otimes m)(t, t') \tag{3.5}$$

From the construction in [41], m is concentrated on a compact and metrizable (hence separable) set of B of $\mathfrak{F}'$, and exists $\{\varphi_n, n \in \mathbb{N}\}$ a dense family of B. Let

$$N = \{(t, t') \in T^2,\ (e_t + e_{t'}, f_t + f_{t'}) \text{ not minimal}\} \tag{3.6}$$

Then by lemma 3.5

$$\forall (t, t') \in N,\ \exists h_{t,t'} > 0,\ (e_t + e_{t'}, f_t + f_{t'}) = \gamma_{t,t'} + (0, \mathrm{h}_{t,t'}) \tag{3.7}$$

Define

$$N_n = \{(t,t') \in N,\ \varphi_n(h_{t,t'}) > 0\} \tag{3.8}$$

Then $N = \bigcup_{n\in\mathbb{N}} N_n$ by the Hahn-Banach theorem.

$$\int_{N_n} (e_t+e_{t'}, f_t+f_{t'}) d(m\otimes m)(t,t') = \int_{N_n} \gamma_{t,t'} d(m\otimes m)(t,t') + \int_{N_n} (0, h_{t,t'}) d(m\otimes m)(t,t') \tag{3.9}$$

and by lemma 3.4 $\int_{N_n}(0,h_t)d(m\otimes m)(t,t') = (0,h)$ is a minimal pair hence $\int_{N_n} h_{t,t'}d(m\otimes m)(t,t') = 0$. It follows that $\int_{N_n}\varphi_n(h_{t,t'})d(m\otimes m)(t,t') = 0$ which implies that $m(N_n) = 0$ since $\varphi_n(h_{t,t'}) > 0$ (This is a classical application of the monotone convergence theorem). Finally by $\sigma-$subadditivity $m(N) = 0$.

4 Applications to invariant kernels and Kreĭn subspaces

As for the direct integral of Kreĭn subspaces, where the Kreĭn space structure only was not sufficient, the kernel alone of a Kreĭn subspace is not sufficient to have a minimal decomposition. This is due to the vector space structure of the set of hermitian kernels, hence the fact the order intervals are not bounded. To get an integral decomposition, we work on Kreĭn-Hilbert pairs and their kernels.
.

4.1 Integral decomposition of invariant minimal Kreĭn-Hilbert pairs

Let $\mathfrak{C} = \mathbf{L}^+(\mathcal{F},\mathcal{E})$ be the cone of positive kernels in $\mathcal{L}(\mathcal{F},\mathcal{E})$ and $\mathfrak{V} = \mathfrak{C} - \mathfrak{C} = \mathbf{L}^b(\mathcal{F},\mathcal{E})$ be the vector space generated by $\mathfrak{C}$, vector space of bounded hermitian kernels. Suppose that the space $\mathcal{L}(\mathcal{F},\mathcal{E})^2$ is weakly complete and conuclear. Then the cone $\digamma = \mathfrak{V}\times\mathfrak{C}$ is a salient closed convex cone, and it has the I.R.P. (this will for notably be the case if $\mathcal{E}$ is the space of distributions on a Lie group G [41]). Let $\mathcal{T}$ be a group of weakly continuous operators on $\mathcal{E}$, and define the following convex cones:

$$\mathbb{U} = \{K \in \mathfrak{V},\ \forall T \in \mathcal{T},\ TKT^* = K\} \tag{4.1}$$

and for $\Lambda = \{\lambda_T,\ T\in\mathcal{T}\}$ a family of positive numbers indexed by $\mathcal{T}$

$$\mathbb{I}_\Lambda = \{H \in \mathfrak{C},\ \forall T\in\mathcal{T},\ THT^* \le \lambda_T H\} \tag{4.2}$$

Finally let $\Gamma_D(\Lambda) = (\mathbb{U}\times\mathbb{I}_\Lambda)\bigcap D$ where D is the cone of dominated pairs.

Lemma 4.1 $\Gamma_D(\Lambda)$ *is a closed convex subcone of* $\digamma$*.*

Proof The convexity is straightforward, and the closedness follows from the weak continuity of $T \subset \mathcal{T}$:
For all $\gamma \in \mathbb{R}$, the operators $\widehat{\gamma T} : \mathcal{L}(\mathcal{F}, \mathcal{E}) \to \mathcal{L}(\mathcal{F}, \mathcal{E})$ defined by $\widehat{\gamma T}(v) = \gamma.v - TvT^*$ are weakly continuous, $\mathbb{U} = \bigcap_{T \in \mathcal{T}} \widehat{1T}^{-1}\{0\}$ and $\mathbb{I}_\Lambda = \bigcap_{T \in \mathcal{T}} \widehat{\lambda_T T}^{-1}\{\mathfrak{C}\}$

Let $(\mathcal{K}, \mathcal{H})$ be an invariant minimal Kreĭn-Hilbert pair of subspaces of $\mathcal{E}$ with kernels (K, H). Using the results of section 2 $\exists \Lambda, (K, H) \in \Gamma_D(\Lambda)$. By the previous theorem if $t \to (K_t, H_t)$, $T \to ext(\Gamma_D(\Lambda))$ is an admissible parametrization of the extreme rays then there exists a Radon measure m on T (unique if the face $\Gamma_D(\lambda)\,((K, H))$ is a lattice) such that

$$(K, H) = \int_T (K_t, H_t) dm(t) \tag{4.3}$$

Define $(\mathcal{K}_t, \mathcal{H}_t)$ the associated family of invariant Kreĭn-Hilbert pairs of subspaces of $\mathcal{E}$ with kernels (K_t, H_t). Then by equation 4.3, this family is pseudo m-integrable and we can define their integral $\int_T (\mathcal{K}_t, \mathcal{H}_t) dm(t)$.

Proposition 4.2 *If the family is m-integrable then*

$$\int_T (\mathcal{K}_t, \mathcal{H}_t) dm(t) = (\mathcal{K}, \mathcal{H}) \tag{4.4}$$

Proof the space $\int_T \mathcal{K}_t dm(t) \subset \int_T \mathcal{H}_t dm(t) = \mathcal{H}$, but $\mathcal{H} = \mathcal{K}$ as subspaces and since $\int_T \mathcal{K}_t dm(t)$ and $\mathcal{K}$ are Kreĭn subspaces, then they coincide (proposition 39 p 246 in [36]).

In fact we have more :

Theorem 4.3 *The integral $\int_T (\mathcal{K}_t, \mathcal{H}_t) dm(t)$ is a direct integral and*

$$(\mathcal{K}, \mathcal{H}) = \int_T^{\oplus} (\mathcal{K}_t, \mathcal{H}_t) dm(t) \tag{4.5}$$

To prove this theorem we need the following lemma

Lemma 4.4 *Let (V, C), (W, D) and (U, P) be three minimal pairs in $\Gamma_D(\Lambda)$ such that $(V, C) = (W, D) + (U, P)$. Suppose moreover that (W, D), (U, P) are extremal. Then either they belong to the same extreme ray or D and P are $\mathfrak{C}$-independent.*

Proof

$$
\begin{aligned}
& C = D + P,\ v = W + U && (4.6)\\
\Rightarrow\ & (C - v) = (D - W) + (P - U),\ (C + v) = (D + W) + (P + U) && (4.7)\\
\Rightarrow\ & (D - W) + (P - U)\ \amalg_{\mathfrak{C}}\ (D + W) + (P + U) && (4.8)
\end{aligned}
$$

Or equivalently in terms of subspaces

$$
\mathcal{D} + \mathcal{P} = ((\mathcal{D} - \mathcal{W}) + (\mathcal{P} - \mathcal{U})) \oplus ((\mathcal{D} + \mathcal{W}) + (\mathcal{P} + \mathcal{U})) \tag{4.9}
$$

Let B_+ be the kernel of the Hilbert subspace $\mathcal{B}_+$, intersection of the Hilbert subspaces with kernel $(D - W)$ and $(P - U)$, and B_- be the kernel of the Hilbert subspace $\mathcal{B}_-$, intersection of the Hilbert subspaces with kernel $(D + W)$ and $(P + U)$. These two kernels are independent, and there sum define a invariant Kreĭn subspace hence $(B = B_+ - B_-, Q = B_+ + B_-) \in \Gamma_D(\Lambda)$. But $(B, Q) \leq_{\Gamma_D(\Lambda)} (W, D); (B, Q) \leq_{\Gamma_D(\Lambda)} (U, P)$, hence either

$$
(B, Q) = \alpha(W, D) = \beta(U, P), \quad \alpha > 0,\ \beta > 0
$$

and (W, D) and (U, P) belong to the same extreme ray or $(B, Q) = 0$ and in this case D and P are $\mathfrak{C}$-independent. (Precisely, in equation 4.9 $\mathcal{D} + \mathcal{P} = (\mathcal{D} - \mathcal{W}) \oplus (\mathcal{P} - \mathcal{U}) \oplus (\mathcal{D} + \mathcal{W}) \oplus (\mathcal{P} + \mathcal{U})$ and the sum $\mathcal{D} \oplus \mathcal{P}$ is direct).

We can now prove theorem 4.3:

Proof From the theory of integral of Hilbert subspaces[42] an integral of Hilbert subspaces is direct if the subspaces are disjoint. But lemma 3.6 combined with lemmas 4.4 and 3.4 prove that the spaces are disjoint. It follows that the integral of Hilbert subspaces is direct, hence that Φ is one-to-one. Finally the integral of Kreĭn-Hilbert pairs of subspaces is direct.

4.2 Extremality and irreducibility

It is easy to prove that a regularly irreducible Kreĭn space with kernel K induces extremal pairs of kernels (K, H) for any minimal majorant H of K (by the regular Schur's lemma 2.12, any self-adjoint projection P is the identity).

However the converse is not true as proves the following example:

Example 4.5 *Consider example 1.10 and $T = J_1$, $\mathcal{E} = \mathcal{K}$. Then (I, J_2) is an extremal pair but $\mathcal{K}$ is reducible ((I, J_1) is not extremal).*

Moreover, it is not clear wether an integral decomposition into irreducible subspaces always exists. The reason is that the set of minimal majorant of K is not bounded as soon as $\mathcal{K}$ is not definite, and maximisation procedures on Choquet's conical measures may fail, as in the following (trivial) example:

Example 4.6 *Consider the same example with $T = I$. Let $P_n^+ = \frac{I-J_n}{2}$ and $P_n^- = \frac{I+J_n}{2}$. then the family $\{J_n,\ n \in \mathbb{N}^*\}$ is not bounded, so are the families of projections $\{P_n^+,\ n \in \mathbb{N}\}$ and $\{P_n^-,\ n \in \mathbb{N}^*\}$, but they define invariant subspaces and $P_n^+ + P_n^- = I$.*

To ensure the regular irreducibility of the pairs of spaces occurring in the decomposition 4.5 we make the following hypothesis (FS):
Fix an invariant Kreĭn space $\mathcal{K}$. Then there exists an isomorphism $J : \mathcal{E} \longrightarrow \mathcal{E}$ such that:

1. (K, JK) is a minimal pair defining $\mathcal{K}$
2. and $\forall L$ in $\mathbb{U}$ (hence verifying equation 4.1), $JLJ^* = L$.

This hypothesis is strong since we have to know the existence of a special symmetry first. However, if the group G is large enough, it may have a representative J such that (K, JK) is a minimal pair (this is what we called a fundamental representation), and in this case the second condition is always fulfilled. It is also the case if the algebra generated by G carries a fundamental symmetry.

Lemma 4.7 *Under the hypothesis (FS), any extremal pair $(L, JL) \in \Gamma_D$ defines a regularly irreducible Kreĭn subspace.*

Proof Suppose the Kreĭn space $\mathcal{L}$ with kernel L and Hilbert majorant $\mathcal{G}$ with kernel $G = JL$ is not regularly irreducible. Then there exists a projection P on $\mathcal{L}$ such that $P(\mathcal{L})$ is an invariant Kreĭn space. Its kernel is obviously PL, direct calculations give that PJL is positive and a minimal majorant of PL since by hypothesis, $PJL = PL(J^*)^{-1} = JPLJ^*(J^*)^{-1} = JPL$. We can do the same for the projection $(I - P)$ and it follows that

$$(L, JL) = (PL, JPL) + ((I - P)L, (J(I - P)L) \tag{4.10}$$

with the three terms in Γ_D, and (L, JL) is not extremal.

Now take for minimal pair of kernels $(K, H = JK)$.

Lemma 4.8 *Suppose* $(K, JK) = (W, D) \oplus (U, E)$. *Then* $D = JW,\ E = JU$.

Proof Let P be the orthogonal projection in the Hilbert subspace with kernel JK on the subspace with kernel D. Then [36] $D = PJK$. But direct calculations give also that $W = PK$ (P is also self-adjoint for the indefinite inner product induced by K).
We can now use the hypothesis:

$$\begin{aligned} D &= PJK = PJKJ^*(J^*)^{-1} \\ &= PK(J^*)^{-1} = W(J^*)^{-1} \\ &= JWJ^*(J^*)^{-1} = JW \end{aligned}$$

Lemma 4.9 *In the decomposition 4.3*

$$(K, H) = \int_T (K_t, H_t) dm(t)$$

we have

$$H_t = JK_t \ m - a.s. \tag{4.11}$$

Proof We know that any measure m verifying equation 4.3 is concentrated on a compact and metrizable (hence separable) set B. Let $\{\varphi_n, n \in \mathbb{N}\}$ be a dense family of B and define

$$N = \{t \in T,\ (K_t, H_t) \neq (K_t, JK_t)\} \tag{4.12}$$

$$N_n^+ = \{t \in N,\ \varphi_n(JK_t - H_t) > 0\} \tag{4.13}$$

$$N_n^- = \{t \in N,\ \varphi_n(JK_t - H_t) < 0\} \tag{4.14}$$

Then $N = \bigcup_{n \in \mathbb{N}} (N_n^+ \cup N_n^-)$ by the Hahn-Banach theorem.
By theorem 4.3, we can change the measure such that the following integral is direct:

$$(K, H) = \int_T^{\oplus} (K_t, H_t) dm(t) = \int_{N_n}^{\oplus} (K_t, H_t) dm(t) \oplus \int_{T \setminus N_n}^{\oplus} (K_t, H_t) dm(t) \tag{4.15}$$

and by lemma 4.8 $\int_{N_n^+} (K_t, H_t) dm(t)$ is of the form (W, JW). But $W = \int_{N_n^+} K_t dm(t)$ hence $JW = \int_{N_n^+} JK_t dm(t)$ and we get $\int_{N_n^+} (JK_t - H_t) dm(t) = 0$. It follows that $\int_{N_n^+} \varphi_n(JK_t - H_t) dm(t) = 0$ which implies that $m(N_n^+) = 0$ since $\varphi_n(JK_t - H_t) > 0$. The same arguments work for N_n^- and finally $m(N) = 0$.

Combining theorem 4.3, lemma 4.9 and lemma 4.7 we get:

Theorem 4.10 *Let $\mathcal{K}$ be a invariant Kreĭn subspace, and suppose that the hypothesis (FS) is verified. Suppose moreover that the space $\mathcal{E}$ is weakly complete and conuclear. Then $(\mathcal{K}, \mathcal{H} = J\mathcal{K})$ admits a direct integral decomposition in terms of irreducible invariant Kreĭn-Hilbert pairs $(\mathcal{K}_t, \mathcal{H}_t = J\mathcal{K}_t)$:*

$$(\mathcal{K}, \mathcal{H}) = \int_T^{\oplus} (\mathcal{K}_t, \mathcal{H}_t)\, dm(t) \tag{4.16}$$

Moreover, we have an analogue of Parseval's formula:

$$\left[k = \int_T^{\oplus} k_t dm(t), k' = \int_T^{\oplus} k_t^{'} dm(t)\right]_{\mathcal{K}} = \int_T \left[k_t, k_t^{'}\right]_{\mathcal{K}_t} dm(t) \tag{4.17}$$

Example 4.11 *Consider example 2.18: $X = \mathbb{R}^3$ and $G = SO(1,2)$ is the associated Lorentz group, $J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ is the Minkowski metric operator.*

Note that $\mathcal{E} = \mathbb{C}^X$ is a nuclear and Frechet space (as a product of nuclear and Frechet spaces [43]), hence conuclear [15], [38].

The pair $K(x,y) = \exp \langle Jx, y\rangle_{\mathbb{R}^3}$, $H(x,y) = \exp \langle x, y\rangle_{\mathbb{R}^3}$ is a fundamental pair of kernels, $K(x,y)$ is invariant under G and for all $g \in G$ $\tau_g(\mathcal{H}) \subset \mathcal{H}$. Moreover $K = \tau_J H$ with J in G, and the representation is fundamental. Finally the hypothesis of theorem 4.10 are fulfilled and $(\mathcal{K}, \mathcal{H} = \tau_J\mathcal{K})$ admits a direct integral decomposition in terms of irreducible invariant Kreĭn-Hilbert pairs $(\mathcal{K}_t, \mathcal{H}_t = \tau_J\mathcal{K}_t)$.

The decomposition is as follows:

$$K_n(x,y) = \frac{(\langle Jx, y\rangle_{\mathbb{R}^3})^n}{n!}, \qquad H_n(x,y) = JK_n(x,y) = \frac{(\langle x, y\rangle_{\mathbb{R}^3})^n}{n!} \tag{4.18}$$

and

$$(\mathcal{K}, \mathcal{H}) = \bigoplus (\mathcal{K}_n, \mathcal{H}_n) \tag{4.19}$$

Note that each Kreĭn space $\mathcal{K}_n$ of homogeneous polynomials of degree n with kernel $K_n(x,y)$ is regularly irreducible by theorem 2.17.

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